

Multipartite fully entangled fraction

Jianwei Xu*

College of Science, Northwest A&F University, Yangling, Shaanxi 712100, China

(Dated: June 16, 2015)

Abstract: Fully entangled fraction is a definition for bipartite states, which is tightly related to bipartite maximally entangled states, and has clear experimental and theoretical significance. In this work, we generalize it to multipartite case, we call the generalized version multipartite fully entangled fraction (MFEF). MFEF measures the closeness of a state to GHZ states. The analytical expressions of MFEF are very difficult to obtain except for very special states, however, we show that, the MFEF of any state is determined by a system of finite-order polynomial equations. Therefore, the MFEF can be efficiently numerically computed.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.65.Aa

I. INTRODUCTION

Quantum entanglement is a crucial ingredient in quantum information processing. In practice, the maximally entangled states are often the ideal resource in many quantum information processing schemes [1, 2]. Fully entangled fraction is a definition for bipartite states, which is tightly related to bipartite maximally entangled states, and has clear experimental and theoretical significance [3–7]. In Ref.[8], the analytical expressions are obtained for the fully entangled fraction of any two-qubit states. In Ref.[9] an upper bound of the fully entangled fraction is obtained. In Ref.[10] some analytical results have been derived for some special states. In Ref.[11], the monogamy relations for multiqubit states via the fully entangled fraction have been investigated. Ref.[12] shows that in $d \otimes d'$ ($2d \leq d'$) system, there exist mixed maximally entangled states. Based on this fact of Ref.[12], Ref.[13] studies the maximally entangled states and fully entangled fraction in general $d \otimes d'$ system.

In this work, we generalize the definition of fully entangled fraction to multipartite case, we call the generalized version multipartite fully entangled fraction (MFEF). Our definition of MFEF is tightly related to the GHZ states. GHZ states are a class of important multipartite entangled states which play vital roles in many experiments either testing the quantum formalism or realizing quantum information processing [2]. The bipartite maximally entangled pure states can be viewed as the bipartite case of GHZ states. Similar to the case of fully entangled fraction, we show that the analytical expressions of MFEF are very difficult to obtain except for very special states. However, we prove that, the MFEF of any state is determined by a system of finite-order polynomial equations. Therefore, the MFEF can be efficiently numerically computed.

This paper is organized as follows. In section 2, we give the definition of MFEF, a lower bound and an upper bound of MFEF, and give the expressions of MFEF for

a class of special pure states. In section 3, we study the MFEF for N -qubit states. In section 4, we show that the MFEF of any state is determined by a system of finite-order polynomial equations, therefore, the MFEF can be efficiently numerically computed. In section 5 we give a summary.

II. THE DEFINITION OF MFEF

Consider the N -partite ($N \geq 2$) system $A_1 A_2 \dots A_N$, its subsystems $\{A_i\}_{i=1}^N$ are all d -dimensional and correspond to the Hilbert space H . We define the multipartite fully entangled fraction (MFEF) of the N -partite state ρ on $H^{\otimes N}$ (we also write $H^{\otimes N} = d^{\otimes N}$ for emphasizing $\dim H = d$) as

$$F(\rho) = \max_{U_1, \dots, U_N} \langle \phi | (\otimes_{i=1}^N U_i^\dagger) \rho (\otimes_{i=1}^N U_i) | \phi \rangle, \quad (1)$$

where \max runs over all local $d \times d$ unitary matrices U_1, U_2, \dots, U_N , $+$ denotes adjoint, and $|\phi\rangle$ is the GHZ state for given orthonormal basis $\{|i\rangle\}_{i=1}^d$ of H ,

$$|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\dots i\rangle. \quad (2)$$

Obviously, the bipartite maximally entangled pure states can be viewed as the bipartite case of GHZ states, and the definition of fully entangled fraction can be viewed as the bipartite case of MFEF. The intuitive meaning of MFEF is that it measures the closeness of a state with respect to the GHZ states.

Theorem 1. The MFEF of the N -partite state ρ on $d^{\otimes N}$ satisfies that

$$\frac{1}{d^N} \leq F(\rho) \leq p_{\max}(\rho) \leq \sqrt{\text{tr}(\rho^2)} \leq 1; \quad (3)$$

$$F(\rho) = 1 \Leftrightarrow \rho \text{ is a GHZ state}; \quad (4)$$

$$F(\rho) = \frac{1}{d^N} \Leftrightarrow \rho = \frac{I^{\otimes N}}{d^N}. \quad (5)$$

In Eq.(3), $p_{\max}(\rho)$ is the maximal eigenvalues of ρ , in Eq.(5), I is the identity operator.

*Electronic address: xxujianwei@nwfau.edu.cn

Proof. We prove this theorem in the similar way of the proof for Theorem 2 in Ref.[10]. Let $\rho = \sum_{i=1}^{d^N} p_i |\psi_i\rangle\langle\psi_i|$ be the eigendecomposition,

$$\begin{aligned} \langle\phi|(\otimes_{i=1}^N U_i^+) \rho (\otimes_{i=1}^N U_i)|\phi\rangle &= \sum_i p_i q_i \\ &\leq p_{\max} \leq \sqrt{\sum_i p_i^2} = \sqrt{\text{tr}(\rho^2)} \leq 1, \end{aligned} \quad (6)$$

with

$$q_i = \langle\phi|(\otimes_{i=1}^N U_i^+) |\psi_i\rangle\langle\psi_i|(\otimes_{i=1}^N U_i)|\phi\rangle. \quad (7)$$

If $\sum_i p_i q_i = 1$, then $p_{\max} = 1, q_{\max} = 1$, thus $\rho = |\psi\rangle\langle\psi|$ is pure, and

$$\langle\phi|(\otimes_{i=1}^N U_i^+) |\psi\rangle\langle\psi|(\otimes_{i=1}^N U_i)|\phi\rangle = 1, \quad (8)$$

it follows that $\rho = (\otimes_{i=1}^N U_i)|\phi\rangle\langle\phi|(\otimes_{i=1}^N U_i^+)$ for any unitary matrices U_1, U_2, \dots, U_N .

Using the method of Lagrange multipliers, it can be shown that the minimum of $\sum_i p_i q_i$ is $\frac{1}{d^N}$ by $p_i = q_i = \frac{1}{d^N}$ for all i , hence Eq.(5) holds. We then end this proof.

Theorem 2. The MFEF of the pure state

$$|\psi\rangle = \sum_{i=1}^d \sqrt{p_i} |ii\dots i\rangle \quad (9)$$

is

$$F(\psi) = \left(\sum_{i=1}^d \sqrt{p_i} \right)^2, \quad (10)$$

where, $\{|i\rangle\}_{i=1}^d$ is an orthonormal basis of H , $p_i \geq 0$, $\sum_{i=1}^d p_i = 1$.

Proof. First note that

$$\begin{aligned} &\sum_{j=1}^d |(U_1)_{ji}(U_2)_{ji}\dots(U_N)_{ji}| \\ &\leq \sqrt{\sum_{j=1}^d |(U_1)_{ji}|^2 \sum_{j=1}^d |(U_2)_{ji}|^2 \dots \sum_{j=1}^d |(U_N)_{ji}|^2} \leq 1. \end{aligned} \quad (11)$$

Let $\{|i\rangle\}_{i=1}^d = \{|j\rangle\}_{j=1}^d = \{|j_1\rangle\}_{j_1=1}^d$ denote the same

orthonormal basis of H , then

$$\begin{aligned} F(\psi) &= \max |\langle\psi|(\otimes_{i=1}^N U_i)|\phi\rangle|^2 \\ &= \frac{1}{d} \max \left| \sum_{ij=1}^d \sqrt{p_i} \langle ii\dots i | \cdot U_1|j\rangle U_2|j\rangle \dots U_N|j\rangle \right|^2 \\ &= \frac{1}{d} \max \left| \sum_{ij,j_1\dots j_N} \sqrt{p_i} \langle ii\dots i | (U_1)_{jj_1} |j_1\rangle \dots (U_N)_{jj_N} |j_N\rangle \right|^2 \\ &= \frac{1}{d} \max \left| \sum_{i=1}^d \sqrt{p_i} \left[\sum_{j=1}^d (U_1)_{ji} (U_2)_{ji} \dots (U_N)_{ji} \right] \right|^2 \\ &\leq \frac{1}{d} \max \left[\sum_{i=1}^d \sqrt{p_i} \left| \sum_{j=1}^d (U_1)_{ji} (U_2)_{ji} \dots (U_N)_{ji} \right| \right]^2 \\ &\leq \frac{1}{d} \left(\sum_{i=1}^d \sqrt{p_i} \right)^2. \end{aligned} \quad (12)$$

Let $U_1 = U_2 = \dots = U_N = I$, we have $|\langle\psi|\phi\rangle|^2 = \frac{1}{d} (\sum_{i=1}^d \sqrt{p_i})^2$. We then end this proof.

III. MFEF OF N -QUBIT STATES

In this section, we give a special N -qubit states which allow analytical MFEF, and investigate the MFEF for arbitrary N -qubit states.

Theorem 3. The MFEF of the two-qubit state

$$\rho = \frac{1}{2^{\otimes N}} (I^{\otimes N} + c \sigma_3^{\otimes N}) \quad (13)$$

is

$$F(\rho) = \frac{1 + |c|}{2^N}, \quad (14)$$

where, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, c is a real number satisfying $|c| \leq 1$.

Proof. For the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (15)$$

and any 2×2 unitary matrix U , there exists a 3×3 real orthogonal matrix O such that [14]

$$U^+ \left(\sum_{j=1}^3 r_j \sigma_j \right) U = \sum_{jk=1}^3 O_{jk} r_k \sigma_j, \quad (16)$$

for any real numbers $\{r_j\}_{j=1}^3$.

For given 2×2 unitary matrices $\{U_l\}_{l=1}^N$, we denote the corresponding real orthogonal matrices as $\{O^{(l)}\}_{l=1}^N$.

For the state in Eq.(13),

$$F(\rho) = \frac{1}{2^N} \{1 + \max[c \langle\phi| \otimes_{l=1}^N (U_l^+ \sigma_3 U_l) |\phi\rangle]\}. \quad (17)$$

Taking Eq.(16) into above equation, with direct computations, we get

$$\begin{aligned} & 2\langle\phi| \otimes_{i=1}^N (U_i^+ \sigma_3 U_i) |\phi\rangle \\ &= O_{33}^{(1)} O_{33}^{(2)} \dots O_{33}^{(N)} + (-1)^N O_{33}^{(1)} O_{33}^{(2)} \dots O_{33}^{(N)} \\ & \quad + (O_{13}^{(1)} + iO_{23}^{(1)}) \dots (O_{13}^{(N)} + iO_{23}^{(N)}) \\ & \quad + (O_{13}^{(1)} - iO_{23}^{(1)}) \dots (O_{13}^{(N)} - iO_{23}^{(N)}). \end{aligned} \quad (18)$$

Consequently,

$$\begin{aligned} & |c\langle\phi| \otimes_{i=1}^N (U_i^+ \sigma_3 U_i) |\phi\rangle| \\ & \leq |c| [|O_{33}^{(1)} \dots O_{33}^{(N)}| + |(O_{13}^{(1)} + iO_{23}^{(1)}) \dots (O_{13}^{(N)} + iO_{23}^{(N)})|] \\ & \leq |c| \cdot \sqrt{\sum_j |O_{j3}^{(1)}|^2} \dots \sqrt{\sum_j |O_{j3}^{(N)}|^2} = |c|. \end{aligned} \quad (19)$$

Conversely, let $O_{13}^{(1)} = O_{13}^{(2)} = \dots = O_{13}^{(N-1)} = 1$, and $O_{13}^{(N)} = \text{sign}(c)$, all above equalities hold. We then end this proof.

We next investigate the MFEF for arbitrary N -qubit states. The 2×2 unitary matrices $\{U^{(l)}\}_{l=1}^N$ can be expressed as

$$U^{(l)} = x_0^{(l)} I + i \sum_{j=1}^3 x_j^{(l)} \sigma_j = \sum_{\mu=0}^3 x_\mu^{(l)} i^{g(\mu)} \sigma_\mu, \quad (20)$$

where $\{x_\mu^{(l)}\}_{\mu=0}^3$ are real numbers,

$$\sum_{\mu=0}^3 (x_\mu^{(l)})^2 = 1, \text{ for all } l, \quad (21)$$

$$\sigma_0 = I, \quad (22)$$

$$g(\mu) = 0 \text{ when } \mu = 0, \quad g(\mu) = 1 \text{ when } \mu = 1, 2, 3. \quad (23)$$

As a result,

$$\begin{aligned} (\otimes_{i=1}^N U_i) |\phi\rangle &= \sum_{\mu_1 \mu_2 \dots \mu_N=0}^3 x_{\mu_1}^{(1)} x_{\mu_2}^{(2)} \dots x_{\mu_N}^{(N)} \\ & \quad \cdot i^{g(\mu_1)+g(\mu_2)+\dots+g(\mu_N)} \sigma_{\mu_1} \otimes \sigma_{\mu_2} \dots \sigma_{\mu_N} |\phi\rangle \end{aligned} \quad (24)$$

$$= \sum_{\mu_1 \mu_2 \dots \mu_N=0}^3 x_{\mu_1}^{(1)} x_{\mu_2}^{(2)} \dots x_{\mu_N}^{(N)} |\phi_{\mu_1 \mu_2 \dots \mu_N}\rangle, \quad (25)$$

where

$$|\phi_{\mu_1 \dots \mu_N}\rangle = i^{g(\mu_1)+\dots+g(\mu_N)} \sigma_{\mu_1} \otimes \dots \otimes \sigma_{\mu_N} |\phi\rangle. \quad (26)$$

Consequently,

$$\begin{aligned} & \langle\phi| (\otimes_{i=1}^N U_i^+) \rho (\otimes_{i=1}^N U_i) |\phi\rangle \\ &= \sum_{\nu_l; \mu_l} x_{\nu_1}^{(1)} \dots x_{\nu_N}^{(N)} x_{\mu_1}^{(1)} \dots x_{\mu_N}^{(N)} \langle\phi_{\nu_1 \dots \nu_N} | \rho | \phi_{\mu_1 \dots \mu_N}\rangle \\ &= \sum_{\nu_l; \mu_l} x_{\nu_1}^{(1)} \dots x_{\nu_N}^{(N)} x_{\mu_1}^{(1)} \dots x_{\mu_N}^{(N)} R_{\nu_1 \dots \nu_N; \mu_1 \dots \mu_N}, \end{aligned} \quad (27)$$

where, $R_{\nu_1 \dots \nu_N, \mu_1 \dots \mu_N}$ is the real part of $\langle\phi_{\nu_1 \dots \nu_N} | \rho | \phi_{\mu_1 \dots \mu_N}\rangle$,

$$\begin{aligned} & R_{\nu_1 \dots \nu_N, \mu_1 \dots \mu_N} \\ &= \frac{1}{2} [\langle\phi_{\nu_1 \dots \nu_N} | \rho | \phi_{\mu_1 \dots \mu_N}\rangle + \langle\phi_{\mu_1 \dots \mu_N} | \rho | \phi_{\nu_1 \dots \nu_N}\rangle], \end{aligned} \quad (28)$$

$$R_{\nu_1 \dots \nu_N, \mu_1 \dots \mu_N} = R_{\mu_1 \dots \mu_N, \nu_1 \dots \nu_N}. \quad (29)$$

Let $\{\lambda_l\}_{l=1}^N$ be the Lagrange multipliers, and let

$$\begin{aligned} L &= \sum_{\nu_l; \mu_l} x_{\nu_1}^{(1)} \dots x_{\nu_N}^{(N)} x_{\mu_1}^{(1)} \dots x_{\mu_N}^{(N)} R_{\nu_1 \dots \nu_N, \mu_1 \dots \mu_N} \\ & \quad - \sum_{l=1}^N \lambda_l \left[\sum_{\mu=0}^3 (x_\mu^{(l)})^2 - 1 \right], \end{aligned} \quad (30)$$

then the extremum conditions yield that

$$\begin{aligned} & \frac{1}{2} \frac{\partial L}{\partial x_{\nu_l}^{(l)}} = \sum_{n \neq \nu_l} x_{\nu_1}^{(1)} \dots x_{\nu_{l-1}}^{(l-1)} x_{\nu_{l+1}}^{(l+1)} \dots x_{\nu_N}^{(N)} x_{\mu_1}^{(1)} \dots x_{\mu_N}^{(N)} \\ & \quad \cdot R_{\nu_1 \dots \nu_N, \mu_1 \dots \mu_N} - \lambda_l x_{\nu_l}^{(l)} = 0, \end{aligned} \quad (31)$$

for all l and all $\nu_l = 0, 1, 2, 3$. Taking Eq.(31) into Eq.(27), we get

$$\langle\phi| (\otimes_{i=1}^N U_i^+) \rho (\otimes_{i=1}^N U_i) |\phi\rangle = \lambda_l. \quad (32)$$

Note that, in general, Eq.(31) leads to many (but finite) solutions of $\{x_{\mu_l}^{(l)}, \lambda_l\}$, we should take the maximum of Eq.(32) for all these λ_l . We conclude this result as Theorem 4 below.

Theorem 4. For any N -qubit state ρ , the real tensor $R_{\nu_1 \dots \nu_N, \mu_1 \dots \mu_N}$ is defined in Eqs.(28,26), then the MFEF of ρ is

$$F(\rho) = \max_{l=1}^N \{ \max[\lambda_l : \text{Eqs.}(31), \text{Eqs.}(21)] \}. \quad (33)$$

We remark that, when $N = 2$, Eq.(33) recovers the two-qubit case in Ref.[8] (noticing that when $N = 2$ we can always equivalently let $U_1 = I$).

IV. MFEF OF N -QUBIT STATES

From the Theorem 4 for the N -qubit ($d = 2$) case, we want to know whether the MFEF of any N -qubit state can also be obtained by solving a system of finite-order polynomial equations. In this section we show this is true. To this aim, we use the generators of $\text{su}(d)$ Lie algebra to represent unitary matrices on H . Consider the d^2 Hermitian operators [15, 16] on H :

$$\{\sigma_\mu\}_{\mu=0}^{d^2-1} = \{\sigma_0 = \sqrt{\frac{2}{d}} I, \sigma_1, \dots, \sigma_{d^2-1}\}, \quad (34)$$

$$\begin{aligned} \{\sigma_i\}_{i=1}^{d^2-1} &= \cup \{(|k\rangle\langle j| + |j\rangle\langle k|)_{j \neq k}\}_{j,k=1}^d \\ & \quad \cup \{-i(|k\rangle\langle j| - |j\rangle\langle k|)_{l < k}\}_{j,k=1}^d \\ & \quad \cup \left\{ \sqrt{\frac{2}{j(j+1)}} (\sum_{k=1}^j |k\rangle\langle k| - j|j+1\rangle\langle j+1|) \right\}_{j=1}^{d-1}. \end{aligned} \quad (35)$$

It is shown that

$$\text{tr}\sigma_0 = \sqrt{2d}, \quad \text{Tr}\sigma_j = 0, \quad j \in \{1, \dots, d^2 - 1\}, \quad (36)$$

$$\text{tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}, \quad \mu, \nu \in \{0, 1, \dots, d^2 - 1\}. \quad (37)$$

For $i, j, k \in \{0, 1, \dots, d^2 - 1\}$,

$$[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{d^2-1} f_{ijk} \sigma_k, \quad (38)$$

$$\{\sigma_i, \sigma_j\} = \frac{4}{d} \delta_{ij} I + 2 \sum_{k=1}^{d^2-1} d_{ijk} \sigma_k, \quad (39)$$

$$\sigma_i \sigma_j = \frac{2}{d} \delta_{ij} I + \sum_{k=1}^{d^2-1} (d_{ijk} + i f_{ijk}) \sigma_k, \quad (40)$$

$$f_{ijk} = \frac{1}{4i} \text{Tr}([\sigma_i, \sigma_j] \sigma_k), \quad (41)$$

$$d_{ijk} = \frac{1}{4} \text{Tr}(\{\sigma_i, \sigma_j\} \sigma_k), \quad (42)$$

where $[\sigma_i, \sigma_j]$, $\{\sigma_i, \sigma_j\}$ are commutator and anticommutator, f_{ijk}, d_{ijk} are called structure constants, f_{ijk} are completely antisymmetric and d_{ijk} completely symmetric. When $d = 2$, $\{\sigma_i\}_{i=1}^3$ are well known Pauli operators, $d_{ijk} = 0$ and f_{ijk} the permutation symbol.

All $d \times d$ complex matrices forms a d^2 -dimensional complex Hilbert space \mathcal{H} equipped the inner product $\langle M_1 | M_2 \rangle = \text{tr}(M_1^\dagger M_2)$ for any $M_1, M_2 \in \mathcal{H}$. Note that Eq.(34) is an orthonormal basis of \mathcal{H} , so any $d \times d$ complex matrix can be expressed in this basis with complex coefficients. We express $d \times d$ Unitary matrix U as

$$U = \sum_{\mu=0}^{d^2-1} z_\mu \sigma_\mu = z_0 \sigma_0 + \sum_{j=1}^{d^2-1} z_j \sigma_j, \quad z_\mu \in \mathbb{C}, \quad (43)$$

where $\{z_\mu\}$ are all complex numbers satisfying the condition $UU^\dagger = I$. Using Eq.(43), after some algebras, $UU^\dagger = I$ leads to

$$\begin{aligned} UU^\dagger &= (|z_0|^2 + \sum_{j=1}^{d^2-1} |z_j|^2) \frac{2}{d} I \\ &+ \sum_{k=1}^{d^2-1} \left[\sqrt{\frac{2}{d}} (z_0^* z_k + z_0 z_k^*) + \sum_{ij=1}^{d^2-1} z_i z_j^* (d_{ijk} + i f_{ijk}) \right] \sigma_k, \end{aligned} \quad (44)$$

thus

$$|z_0|^2 + \sum_{j=1}^{d^2-1} |z_j|^2 = \frac{d}{2}, \quad (45)$$

$$\sqrt{\frac{2}{d}} (z_0^* z_k + z_0 z_k^*) + \sum_{ij=1}^{d^2-1} z_i z_j^* (d_{ijk} + i f_{ijk}) = 0. \quad (46)$$

Let $z_\mu = x_\mu + i y_\mu$ with x_μ, y_μ real, and notice that f_{ijk} are completely antisymmetric and d_{ijk} completely sym-

metric, then Eqs.(45,46) become

$$\sum_{\mu=0}^{d^2-1} (x_\mu^2 + y_\mu^2) - \frac{d}{2} = 0, \quad (47)$$

$$2\sqrt{\frac{2}{d}} (x_0 x_k + y_0 y_k) + \sum_{ij=1}^{d^2-1} [(x_i x_j + y_i y_j) d_{ijk} + 2x_i y_j f_{ijk}] = 0. \quad (48)$$

Let $z_{\mu_l}^{(l)} = x_{\mu_l}^{(l)} + i y_{\mu_l}^{(l)}$ be the complex numbers corresponding to $U^{(l)}$ as in Eqs.(47,48), then

$$\begin{aligned} &\langle \phi | (\otimes_{l=1}^N U_l^\dagger) \rho (\otimes_{l=1}^N U_l) | \phi \rangle \\ &= \sum_{\nu_1, \mu_1} z_{\nu_1}^{(1)*} \dots z_{\nu_N}^{(N)*} z_{\mu_1}^{(1)} \dots z_{\mu_N}^{(N)} \langle \phi_{\mu_1 \mu_2 \dots \mu_N} | \rho | \phi_{\mu_1 \mu_2 \dots \mu_N} \rangle \end{aligned} \quad (49)$$

where

$$|\phi_{\mu_1 \mu_2 \dots \mu_N}\rangle = \sigma_{\mu_1} \otimes \dots \sigma_{\mu_N} |\phi\rangle. \quad (50)$$

Let

$$\begin{aligned} L &= \langle \phi | (\otimes_{l=1}^N U_l^\dagger) \rho (\otimes_{l=1}^N U_l) | \phi \rangle \\ &- \sum_{l=1}^N \lambda_l \left\{ \sum_{\mu_l=0}^{d^2-1} [(x_{\mu_l}^{(l)})^2 + (y_{\mu_l}^{(l)})^2] - \frac{d}{2} \right\} \\ &- \sum_{l=1}^N \sum_{k=1}^{d^2-1} \tau_{lk} \left\{ 2\sqrt{\frac{2}{d}} (x_0^{(l)} x_k^{(l)} + y_0^{(l)} y_k^{(l)}) \right. \\ &\left. + \sum_{ij=1}^{d^2-1} [(x_i^{(l)} x_j^{(l)} + y_i^{(l)} y_j^{(l)}) d_{ijk} + 2x_i^{(l)} y_j^{(l)} f_{ijk}] \right\}, \end{aligned} \quad (51)$$

then

$$\frac{\partial L}{\partial x_{\mu_l}^{(l)}} = 0, \quad \frac{\partial L}{\partial y_{\mu_l}^{(l)}} = 0, \quad \frac{\partial L}{\partial \lambda_l} = 0, \quad \frac{\partial L}{\partial \tau_{lk}} = 0, \quad (52)$$

constitute a system of finite-order polynomial equations in variables $\{x_{\mu_l}^{(l)}, y_{\mu_l}^{(l)}, \lambda_l, \tau_{lk}\}$. Once we get $\{x_{\mu_l}^{(l)}, y_{\mu_l}^{(l)}, \lambda_l, \tau_{lk}\}$ from Eqs.(52), taking them into $\langle \phi | (\otimes_{l=1}^N U_l^\dagger) \rho (\otimes_{l=1}^N U_l) | \phi \rangle$, we can get the MFEF. Note that, in general, Eqs.(52) leads to many (but finite) solutions of $\{x_{\mu_l}^{(l)}, y_{\mu_l}^{(l)}, \lambda_l, \tau_{lk}\}$, we should take the maximum of $\langle \phi | (\otimes_{l=1}^N U_l^\dagger) \rho (\otimes_{l=1}^N U_l) | \phi \rangle$ for all these $\{x_{\mu_l}^{(l)}, y_{\mu_l}^{(l)}, \lambda_l, \tau_{lk}\}$.

V. SUMMARY

We generalized the definition of bipartite fully entangled fraction to the multipartite case, we called the generalized version multipartite fully entangled fraction (MFEF). MFEF is defined with respect to the multipartite GHZ states then it measures the closeness of a state to GHZ states. We gave two classes of states which allow

analytical MFEF, explored the bounds of MFEF. For N -qubit states, the optimization of MFEF is relatively simple, we provided a calculation scheme. Although the analytical MFEF are very hard to get for general states, we proved that, the MFEF of any state can be efficiently numerically computed.

This work was supported by the Chinese Universities Scientific Fund (Grant No.2014YB029) and the National Natural Science Foundation of China (Grant No.11347213). The author thanks Kai-Liang Lin for helpful discussions.

-
- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
 - [2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009) and references therein.
 - [3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
 - [4] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).
 - [5] S. Albeverio, S. M. Fei, and W. L. Yang, Phys. Rev. A **66**, 012301 (2002).
 - [6] Z. W. Zhou and G. C. Guo, Phys. Rev. A **61**, 032108 (2000).
 - [7] S. K. Özdemir, K. Bartkiewicz, Y. X. Liu, and A. Miranowicz, Phys. Rev. A **76**, 042325 (2007).
 - [8] J. Grondalski, D. M. Etlinger, and D. F. V. James, Phys. Lett. A **300**, 573 (2002).
 - [9] M. Li, S. M. Fei, and Z. X. Wang, Phys. Rev. A **78**, 032332 (2008).
 - [10] M. J. Zhao, Z. G. Li, S. M. Fei, and Z. X. Wang, J. Phys. A: Math. Theor. **43**, 275203 (2010).
 - [11] S. Lee and J. Park, Phys. Rev. A **79**, 054309 (2009).
 - [12] Z. G. Li, M. J. Zhao, S. M. Fei, H. Fan, and W. M. Liu, Quantum Inf. Comp. **12**, 0063 (2012).
 - [13] M. J. Zhao, Phys. Rev. A **91**, 012310 (2015).
 - [14] J. F. Cornwell, *Group Theory in Physics*, vol. I, (Academic Press, New York, 1984) (Chapter 3, Section 5).
 - [15] F. T. Hioe and J. H. Eberly, Phys. Rev. Lett. **47**, 838 (1981).
 - [16] J. Schlienz, G. Mahler, Phys. Rev. A **52**, 4396 (1995).